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Evaluation of Glueball Masses from Lattice Gauge Theories and Scaling Behavior

Silvia Necco[1] **and Rainer Sommer**[1,2]

[1] DESY Platanenallee, 6
15738 Zeuthen, Germany
E-mail: silvia.necco@desy.de

[2] CERN-TH
1211 Geneva 23, Switzerland
E-mail: rainer.sommer@desy.de

We present the non-perturbative evaluation of the glueball masses in the framework of gauge fields formulated on a Euclidean 4-dimensional lattice. We introduce the basic concepts of lattice gauge theories and the general ideas to compute physical observables from Monte Carlo simulations. We then concentrate our attention on the extraction of the glueball masses from exponential decay of correlation functions and on the investigation of the scaling behavior for different lattice formulations.

1 Introduction

The strong, weak and electromagnetic interactions are up to now described successfully within the Standard Model of elementary particles. This theory is based on a *local gauge principle*, with the gauge group

$$G_{loc} = \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y, \tag{1}$$

and it is essentially determined once the matter fields and their transformation laws under G_{loc} are specified.
The degrees of freedom are respectively the color for $\mathrm{SU}(3)$, weak isospin for $\mathrm{SU}(2)$ and weak hypercharge for $\mathrm{U}(1)$.
In particular, quantum chromodynamics (QCD) associated to the color group $\mathrm{SU}(3)$ is the currently accepted framework to describe strong interactions. The matter components are the quarks, which are described by spinor fields carrying three color indices and appearing in six flavors (up, down, charm, strange, top and bottom). The gluons are the vector bosons that mediate the interactions and, due to the fact that the gauge group $\mathrm{SU}(3)$ is non-Abelian, self-interactions exist, contrary to the situation in electromagnetism.
The two main features of strong interactions are asymptotic freedom and confinement, which are related to their properties in the high and low energy regimes, respectively. Asymptotic freedom means that the constituents behave as if they were free particles at very high energies: the running coupling becomes small in this regime and one expects that perturbative methods furnish reliable predictions for physical observables.
On the other hand, confinement corresponds to the fact that quarks have never been detected in isolation, but only as constituents of hadrons. From the theoretical point of view, this should correspond to the fact that all physical states are singlets with respect to the color group. In order to check whether this feature is contained in QCD, one can not apply

perturbative methods, since the coupling is expected to be large at scales corresponding to the size of hadrons. It is however widely believed that this behavior is a consequence of quantum chromodynamics, although up to now no proof exists.
One of the strongest evidence for this to be true comes from lattice field theory. In 1974 Wilson proposed[1] a formulation of a gauge field theory on a discretized Euclidean space-time.
The lattice formulation is then one of the most elegant and powerful non-perturbative methods. An important advantage of this formulation is that the expectation values of physical observables can be computed numerically via numerical simulations; the possibility to evaluate observables *from first principles* is hence an opportunity to test whether QCD provides the the correct framework for describing strong interactions.
However, there are practical limitations, mainly due to the fact that the accessible lattice volumes and resolutions are restricted by the available (finite) computer performance and memory.
In this report we will present the basic concepts of lattice gauge theories and numerical simulations; then we will concentrate on the calculation of glueball masses and on the comparison between different lattice formulations.

2 Lattice Gauge Fields

2.1 Wilson Action

The starting point for a lattice formulation of quantum field theory is the path integral formalism in Euclidean space-time. In this context it is possible to establish a useful analogy between Euclidean quantum field theory on a lattice and statistical mechanics, which is also the basis for numerical simulations. In the following we will consider pure $\mathrm{SU(N)}$ gauge theories on a 4 dimensional hypercubic lattice

$$\Lambda = a\mathbb{Z}^4 = \{x | x_\mu / a \in \mathbb{Z}\}, \quad \mu = 0, 1, 2, 3, \tag{2}$$

where a is the lattice spacing. The lattice introduces an *ultraviolet cutoff* $\propto 1/a$ and provides the only known consistent non-perturbative regularization of non-Abelian gauge theories.

An $\mathrm{SU(N)}$ lattice gauge field is an assignment of a matrix $U(x,\mu) \in \mathrm{SU(N)}$ to every lattice bond with endpoints x and $(x + a\hat{\mu})$, where $\hat{\mu}$ denotes the unit vector in the positive μ direction. A particular gauge-invariant object one can construct on the lattice is the trace of the product of link variables along a closed curve. These loops can be of arbitrary size and shape, and can be taken to lie in any representation of $\mathrm{SU(N)}$. The simplest example is the plaquette $W^{1\times1}_{\mu\nu}$, a (1×1) loop (fig. 1)

$$W^{1\times1}_{\mu\nu}(x) = \mathrm{Tr}\,\left\{U(x,\mu)U(x + a\hat{\mu},\nu)U^{-1}(x + a\hat{\nu},\mu)U^{-1}(x,\nu)\right\}. \tag{3}$$

The action which has been proposed by Wilson[1] for pure lattice gauge theory is defined in terms of these plaquette variables

$$S = \beta \sum_x \sum_{\mu<\nu} \left\{1 - \frac{1}{N}\mathrm{Re}\, W^{1\times1}_{\mu\nu}(x)\right\}. \tag{4}$$

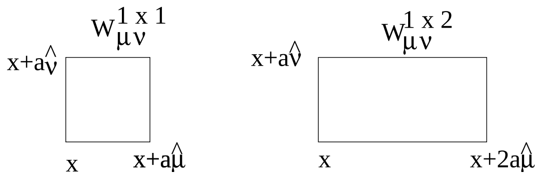

Figure 1. Loops contributing to the Wilson ($W_{\mu\nu}^{1\times 1}$) and RG actions ($W_{\mu\nu}^{1\times 1}$ and $W_{\mu\nu}^{1\times 2}$).

One can show that the leading term of eq. 4 for small a coincides with the continuum Yang-Mills Euclidean action

$$S_{YM} = -\frac{1}{2g_0^2}\int d^4x \,\mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu} = \frac{1}{4}\int d^4x F^a_{\mu\nu}F^a_{\mu\nu}, \tag{5}$$

if we set $\beta = \frac{2N}{g_0^2}$ and identify g_0 with the *bare coupling constant* of the lattice theory; thus the Wilson action gives the desired classical continuum limit.
After having defined the field variables and the action, the next step to quantization is to specify the functional integral. Given an observable $\mathcal{O}[U]$ which is in general a gauge invariant function of the link variables, its expectation value is given by

$$\langle\mathcal{O}\rangle = \frac{1}{Z}\int D[U]\mathcal{O}[U]e^{-S[U]}, \tag{6}$$

where

$$Z = \int D[U]e^{-S[U]} \tag{7}$$

and $D[U] = \prod_{x,\mu} dU(x,\mu)$ is the invariant group measure or *Haar measure*. In analogy to statistical mechanics, Z is called *partition function*.

2.2 Continuum Limit and Improvement

Once one has evaluated a certain observable in lattice gauge theories, it remains to understand how this is related to the continuum physical world. In order to obtain the theory in the continuum, the lattice spacing has to be sent to zero; at the same time, the cut-off goes to infinity and one has to construct *renormalized* physical quantities which remain finite in the continuum limit. The *renormalization group equation* describes how the parameters behave by changing the scale of the theory (in this case the lattice spacing).
If the lattice spacing a is small enough, one expects that dimensionless ratios of physical quantities appear to be nearly independent on a; in this case one speaks of *scaling*. The size of the corrections, or *scaling violations*, will in general depend on which quantities are being considered.
In practice one has to compute the quantities of interest at different values of the lattice spacing a and to extrapolate the results to $a = 0$. An obvious limitation in this procedure is the fact that it is not possible to perform numerical simulations at arbitrarily small lattice spacings. In current calculations, one reaches $a \sim 0.05\mathrm{fm}$ for pure gauge simulations and the quenched approximation, and $a \sim 0.1\mathrm{fm}$ for full QCD.
Then the *improvement* of the lattice formulation turns out to be a very important topic, and

in the last years there were a lot of efforts in this direction, both for the gauge and the fermionic action. The basic idea is to add appropriate combinations of *irrelevant* operators to the lattice action and to tune their coefficients such that the discretization errors are reduced. Using the language of statistical mechanics, the concept of *universality* ensures that the different formulations lead to the same physical continuum limit. At finite lattice spacing, the irrelevant operators govern the discretization errors of renormalized dimensionless quantities.
Several approaches were studied with this purpose; in particular we concentrated on the so called Renormalization Group (RG) lattice gauge actions. The most popular examples are the Iwasaki[2] and DBW2[3] gauge actions; in both cases, in addition to the usual plaquette term, planar rectangular (1×2) loops are included (fig. 1)

$$S = \beta \sum_x \left(c_0 \sum_{\mu<\nu} \left\{ 1 - \frac{1}{3} \mathrm{Re}\, W^{1\times1}_{\mu\nu}(x) \right\} + c_1 \sum_{\mu,\nu} \left\{ 1 - \frac{1}{3} \mathrm{Re}\, W^{1\times2}_{\mu\nu}(x) \right\} \right), \tag{8}$$

with the normalization condition $c_0 = 1 - 8c_1$.
The coefficient c_1 in eq. 8 takes the values

$$c_1 = \begin{cases} -0.331 & \text{Iwasaki} \\ -1.4088 & \text{DBW2} \end{cases} \tag{9}$$

RG actions have been used in recent years for an advanced computation of the light hadron spectrum[4] and have been suggested by some collaboration[5] as good candidates to be used in the next simulations with dynamical fermions. Before one starts to use extensively these alternative actions, it is important to investigate their properties, starting by checking the fundamental one which is the universality. Moreover, the scaling behavior has to be tested for a possibly large number of observables in order to establish how efficient is the improvement. In a previous project we evaluated in particular the critical temperature for deconfinement T_c for DBW2 and Iwasaki action[6] and we confirmed the universality between the Wilson and Iwasaki action. Moreover, the scaling behavior of the Iwasaki action was found to be better than the one for the Wilson action. On the other hand, the results for the DBW2 action showed larger lattice artefacts.

2.3 Numerical Simulation

The goal of numerical simulations in lattice gauge theories is to estimate the expectation values of eq. 6. These integrals involve a very large number of variables, so that the only possibility to evaluate them is to use a so called *Monte Carlo* integration.
An efficient way to proceed is to generate field configurations with a probability distribution which follows the *Boltzmann factor* $e^{-S[U]}$. This method is called *importance sampling* and allows to generate the configurations which have the most substantial weight in the path integral. In analogy to statistical mechanics, one defines an *ensemble of configurations* as an infinite number of field configurations, with a probability density $W[U]$. The probability density associated to the *canonical ensemble* is proportional to the Boltzmann factor

$$W_c[U] \propto e^{-S[U]}. \tag{10}$$

The purpose of a numerical simulation is to generate *samples* consisting of a large number N of configurations $\{[U_n], n = 1, ..., N\}$ such that the distribution within a sample reproduces the desired distribution in the canonical ensemble. The *sample average* of a certain observable $\mathcal{O}$ is defined as

$$\overline{\mathcal{O}} = \sum_{n=1}^{N} \mathcal{O}[U_n] \tag{11}$$

and is an estimator of the ensemble average which corresponds to the expectation value $\langle \mathcal{O} \rangle$.
An *updating* is then a stochastic process which creates the sequence $\{[U_n], n = 1, ..., N\}$. The transition $[U] \to [U']$ happens with a given probability $P([U'] \leftarrow [U])$. In general one requires that the following properties are satisfied:

1. $\int D[U']P([U'] \leftarrow [U]) = 1$;
2. strong ergodicity: each field configuration can be reached from any other one with a finite probability.
3. normalization condition: $\int D[U]W[U] = 1$.

Moreover, one has to require that, starting from a configuration with density W_0, after applying a certain number of updating steps one reaches the equilibrium distribution:

$$\lim_{k \to \infty} P^k W_0 = W_c, \tag{12}$$

from which it follows that $PW_c = W_c$.
In particular, for our numerical simulations we adopted an updating algorithm composed by *heatbath*[7] and *overrelaxation*[8] steps.
In order to evaluate correctly the statistical errors assigned to the observables, one usually makes use of the so-called *jackknife binning*. Recently[9] a method to estimate explicitly the relevant autocorrelation functions and times has been proposed as alternative to the binning procedure.

3 Scaling Properties of RG Actions: The Glueball Masses

Glueballs represent one of the most fascinating prediction that can be obtained in the gauge sector of QCD. They are bound states originating from the self-coupling between gluons.
Osterwalder and Seiler showed in 1978[10] that in strongly coupled lattice gauge theory the lowest eigenstate of the Hamiltonian above the vacuum has a mass m, which is usually called *mass gap*. This result can not be obtained in the framework of perturbation theory; for this reason, Monte Carlo simulations on the lattice provide a powerful tool to evaluate the glueball masses; they were indeed among the first quantities that have been computed on the lattice[11].
Since the theoretical discovery of the glueballs, also the experimental search started[12]. The difficult task in the experimental search is to distinguish the glueball states from the background of mesons (quark-antiquark states), which have the same quantum numbers. For this reason the *exotic* glueballs are particularly interesting because due to their quantum numbers they can not mix with conventional meson states.

Iwasaki action				DBW2 action			
β	L	n_l	n_{meas}	β	L	n_l	n_{meas}
2.2423	10	2,4,6,8	12000	0.8342	12	2,4,6,8	8000
2.2879	12	2,4,6,8	16000	0.9636	16	3,6,9,12	2500
2.5206	16	3,6,9,12	8000				

Table 1. Simulation parameters for the evaluation of the glueball masses for the Iwasaki and DBW2 actions. L indicates the number of lattice points in all four directions, n_l the smearing levels and n_{meas} the number of measurements.

Lattice QCD investigations[13] also addressed the effects of dynamical quarks and glueball-meson mixing on the glueball spectrum from lattice QCD.

Apart from the physical relevance, the mass of the lightest (0^{++}) glueball is particularly interesting since several calculations with the Wilson action[14–17] showed large lattice artefacts of about 40% at coarse lattice spacings $a \simeq 0.15\mathrm{fm}$ and still 20% at $a \simeq 0.10\mathrm{fm}$. This quantity could hence in principle provide a stringent test on the scaling behavior and universality of alternative gauge actions.

3.1 Glueball States

The glueball states are conventionally labeled by

$$|\Psi\rangle = |J^{PC}\rangle, \tag{13}$$

where J represents the spin, the parity $P = \pm 1$ is the eigenvalue of the space reflection and $C = \pm 1$ is the eigenvalue of the charge conjugation.

In the continuum Euclidean space the spin of a state is characterized by the unitary irreducible representations of $SO(3)$. On a cubic lattice the rotation symmetry is broken down to the cubic group O; the physical states have hence to be classified according to the unitary irreducible representations of O.

The cubic group contains 24 elements, corresponding to the permutations of the four space diagonals of a cube. There are five irreducible representations, denoted by A_1, A_2, E, T_1 and T_2, which have dimensions 1,1,2,3 and 3. Since the cubic group is a subgroup of $SO(3)$, any representation D_J of $SO(3)$ for a state of spin J will induce a representation on the group O, the so-called subduced representation D_J^O. It will in general no longer be irreducible and can be decomposed into the irreducible representations of O. Up to $J = 2$ one finds

$$D_0^O = A_1 \tag{14}$$
$$D_1^O = T_1 \tag{15}$$
$$D_2^O = E \oplus T_2. \tag{16}$$

For example, a spin 2 particle is described in the continuum by a quintuplet of degenerate states; on the lattice the quintuplet is split into a doublet E and a triplet T_2. At finite lattice spacing one expects a mass splitting between the two representations, so that $m_E \neq m_{T_2}$. By approaching the continuum limit the ratio m_E/m_{T_2} is expected to converge to one in order to restore the full Euclidean rotational symmetry.

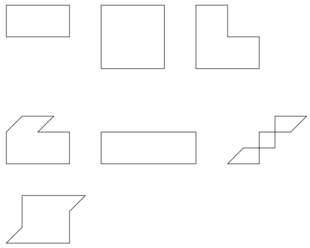

Figure 2. The loopshapes that have been used for the computation of glueball correlation functions.

3.2 Extraction of Glueball Masses from Euclidean Correlation Functions

We decided to concentrate our attention on the 0^{++} and 2^{++} states by measuring the masses in the representations $\mathcal{R} = A_1^{++}, E^{++}, T_2^{++}$.
We measured the connected correlation functions [a]

$$C^{\mathcal{R}}(t) = \langle \mathcal{O}^{\mathcal{R}}(t)\mathcal{O}^{\mathcal{R}}(0)\rangle - \langle \mathcal{O}^{\mathcal{R}}(t)\rangle\langle \mathcal{O}^{\mathcal{R}}(0)\rangle, \tag{17}$$

where $\mathcal{O}^{\mathcal{R}}(t)$ are gauge invariant operators defined at a given time t.
The simplest choice is the space-like plaquettes $W_{ij}^{1\times 1}(x)$ of eq. 3, with $i, j = 1, 3$. More generally, one can consider other loops contained in the space-like directions; after investigating the signal-to-noise ratio of all loops up to length 8 -for which all irreducible representations of the cubic group have been constructed[18]- we decided to consider the $N = 7$ loops shown in fig. 2.
Moreover, we adopted *smearing*[19] techniques to enhance the overlap with the physical state that we wanted to measure. This techniques consists in constructing for different smearing levels $l = 0, ..., M-1$ *smeared* spatial links

$$U_l(x, k) = \mathcal{S}^{n_l} U(x, k), \tag{18}$$

where $\mathcal{S}$ is the so-called *smearing operator*.
Then the correlation function eq. 17 becomes in fact a correlation *matrix* $C_{lm}^{\mathcal{R}}(t)$ with indices $l, m = 1, ..., N \cdot M$. We expect that

$$C_{lm}^{\mathcal{R}}(t) = \langle \Psi_k^R | e^{-Ht} | \Psi_l^R \rangle = \sum_\alpha \langle \Psi_k^R | \Psi_\alpha \rangle \langle \Psi_\alpha | \Psi_l^R \rangle e^{-m_\alpha(R)t}, \tag{19}$$

where $|\Psi_\alpha\rangle$ are eigenstates of the Hamiltonian associated to the system. At large t the lowest mass $m_0(R)$ dominates and it is identified with a glueball state which in the continuum limit will is expected to have the lowest spin contained in the representation R.

As already pointed out since the first works, the calculation of glueball masses presents a lot of technical difficulties; due to the fact that these are quite heavy ($m_G \geq 1.6\mathrm{GeV}$), the signal in the correlation functions of the gluonic excitations decay fast and disappears in the noise.
For the RG actions there is however an additional problem related to the violation of physical positivity[20], which is a consequence of adding irrelevant operators to the Wilson gauge action. Investigations of this feature[6] showed that the extraction of glueball masses must

[a] Note that the subtraction of the connected part is necessary only for $\mathcal{R} = A_1^{++}$, since it has the same quantum numbers as the vacuum.

be considered with great care, and that the measurements can be affected by large uncontrolled systematic errors. One expects that this effect disappears starting from a certain t_{min}, and hence the glueball mass must be extracted from the correlation function at a $t \gg t_{min}$, where $t_{min} \sim 1a - 2a$ for the RG actions.
The simulation parameters are reported in tab. 1. The updating programs have been written in the language FORTRAN 90 for a CRAY T3E machine; the starting point was a parallelized code used in a previous project. For the evaluation of the glueballs we used relatively small lattices (up to 16^4) for which we adopted a single-processor version of the program; we implemented trivial parallelization by simulating different replica of the lattice on several processors in order to improve the statistics of our measurements.

4 Results

The glueball masses in lattice units were extracted from the correlation matrices eq. 17 by employing variational techniques[21].
Concerning the determination of $m_{2^{++}}$ we observed that the signal for the E^{++} channel is usually worse than for the T_2^{++} and the errors on the effective masses are very large already at $t = 3a$. For this reason we decided to use $m_{T_2^{++}}$ as estimate of $m_{2^{++}}$ at finite lattice spacing.
In order to investigate the scaling behavior, one has to build renormalized dimensionless quantities. In particular we have chosen the length $r_0 \sim 0.5\mathrm{fm}$[22], which has been evaluated in lattice units for RG actions at our specific values of β[6]. The renormalized quantities $r_0 m_G$ are reported in tab. 3. Fig. 3 shows the results for $r_0 m_{0^{++}}$ and $r_0 m_{2^{++}}$ as function of the lattice spacing $(a/r_0)^2$. For the comparison we included the results for the *fixed point* (FP) action[23] and several calculations performed with the Wilson action.
The continuum values avaliable in the literature are listed in tab. 2; by using $r_0 = 0.5\mathrm{fm}$, one obtains $m_{0^{++}} \simeq 1.6 - 1.7\mathrm{GeV}$.
Altought the errors are very large, we want to stress that our determination can be seen at least as an upper limit for $m_{0^{++}}$ and $m_{2^{++}}$. From our investigations, we expect that at the values of t/a (usually between 3 and 4) at which we extracted the masses, the effects of positivity violations have already disappeared. For this reason we believe that possible systematic uncertainties on the glueball masses could only be due to the presence of excited states and hence could affect our measurement only in such a way that the real values of $m_{0^{++}}, m_{2^{++}}$ are lower with respect to our determination.
For the 0^{++} channel, at lattice spacings $a \sim 0.15\,\mathrm{fm}$ we notice an improvement of the RG actions with respect to the Wilson action; comparing with the continuum limit we find no significant discrepancy both for DBW2 and Iwasaki action, while for the Wilson action one finds $30 - 40\%$ deviation.
At lattice spacing $a \sim 0.1\,\mathrm{fm}$ we find on the other hand large lattice artefacts for RG actions: the result obtained with the Iwasaki action is practically compatible with the one calculated through the plaquette action at the same lattice spacing, while for the DBW2 action it is even further away from the continuum limit.
If one considers our measurement as upper limit, one could conclude that the RG improved actions are not able to cure the problem of large lattice artefacts for the 0^{++} glueball mass. With our results it is not possible to investigate how the continuum limit is approached, because one should evaluate the glueball masses at smaller lattice spacing and this was

Collab.	$r_0 m_{0^{++}}$	$r_0 m_{2^{++}}$
M & P[24]	4.21(11)(4)	5.85(2)(6)
GF11[25]	4.33(10)	6.04(18)
Teper[26]	4.35(11)	6.18(21)
UKQCD[14]	4.05(16)	5.84(18)
FP[23]	4.12(21)	[5.96(24)]

Table 2. Continuum extrapolations of the two lowest glueball masses in units of r_0. For the FP action, the 2^{++} value is not extrapolated to the continuum but denotes the mass obtained at a lattice spacing $a = 0.10\,\mathrm{fm}$.

beyond the purpose of this work. Regarding $r_0 m_{2^{++}}$, the calculation performed with the Wilson action do not show significant lattice artefacts. At our smallest lattice spacing we do not observe a deviation from the results obtained with the Wilson action; one has however to notice that our errors are too large to make any conclusive statement.

Our computation of glueball masses for RG actions, together with our previous evaluation on the critical temperature investigated the scaling behavior and pointed out possible difficulties connected to the violation of physical positivity. These results will furnish a useful guideline in the choice of the lattice gauge action for future simulations.

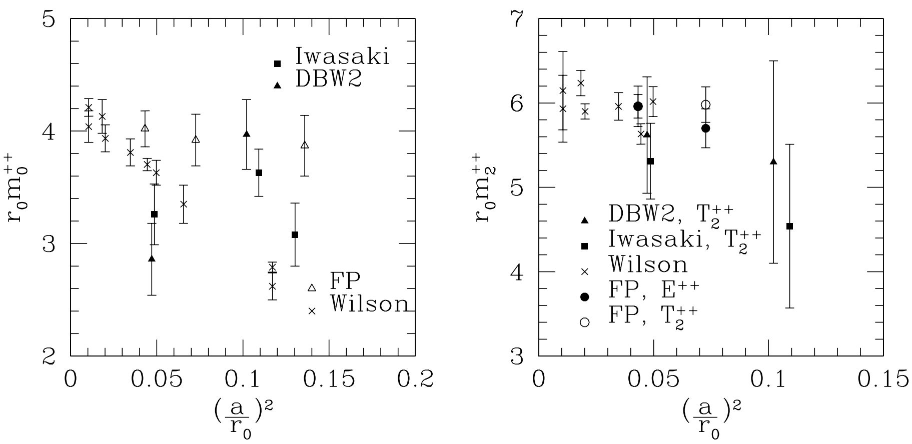

Figure 3. The 0^{++} and 2^{++} glueball masses normalized with r_0 as function of $(a/r_0)^2$ for different actions.

Iwasaki action			DBW2 action		
β	$r_0 m_{A_1^{++}}$	$r_0 m_{T_2^{++}}$	β	$r_0 m_{A_1^{++}}$	$r_0 m_{T_2^{++}}$
2.2423	3.08(28)				
2.2879	3.63(21)	4.54(97)	0.8243	3.97(31)	5.3(1.2)
2.5206	3.26(27)	5.31(45)	0.9636	2.86(32)	5.62(69)

Table 3. Results for $r_0 m_G$ for the channels A_1^{++} and T_2^{++}, using the Iwasaki and DBW2 action.

References

1. Kenneth G. Wilson. Confinement of quarks. *Phys. Rev.*, D10:2445–2459, 1974.
2. Y. Iwasaki. Renormalization group analysis of lattice theories and improved lattice action. 2. four-dimensional nonabelian su(n) gauge model. UTHEP-118.
3. T. Takaishi. Heavy quark potential and effective actions on blocked configurations. *Phys. Rev.*, D54:1050–1053, 1996.
4. A. Ali Khan et al. Light hadron spectroscopy with two flavors of dynamical quarks on the lattice. *Phys. Rev.*, D65:054505, 2002.
5. Y. Aoki et al. Domain wall fermions with improved gauge actions. 2002.
6. Silvia Necco. Universality and scaling behavior of rg gauge actions. 2003.
7. M. Creutz. Monte carlo study of quantized su(2) gauge theory. *Phys. Rev.*, D21:2308–2315, 1980.
8. M. Creutz. Overrelaxation and monte carlo simulation. *Phys. Rev.*, D36:515, 1987.
9. U. Wolff. Monte Carlo errors with less errors. 2003.
10. K. Osterwalder and E. Seiler. Gauge field theories on the lattice. *Ann. Phys.*, 110:440, 1978.
11. B. Berg. Plaquette-plaquette correlations in the su(2) lattice gauge theory. *Phys. Lett.*, B97:401, 1980.
12. S. Godfrey and J. Napolitano. Light meson spectroscopy. *Rev. Mod. Phys.*, 71:1411–1462, 1999.
13. C. McNeile and C. Michael. Mixing of scalar glueballs and flavour-singlet scalar mesons. *Phys. Rev.*, D63:114503, 2001.
14. G. S. Bali et al. A comprehensive lattice study of su(3) glueballs. *Phys. Lett.*, B309:378–384, 1993.
15. C. Michael et al. The glueball spectrum in su(3). *Nucl. Phys.*, B314:347, 1989.
16. P. de Forcrand et al. The 0++ glueball mass in su(3) lattice gauge theory: Towards definitive results. *Phys. Lett.*, B152:107, 1985.
17. H. Chen et al.. The scalar and tensor glueballs in the valence approximation. *Nucl. Phys. Proc. Suppl.*, 34:357–359, 1994.
18. B. Berg and A. Billoire. Glueball spectroscopy in four-dimensional su(3) lattice gauge theory. 1. *Nucl. Phys.*, B221:109, 1983.
19. M. Albanese et al. Glueball masses and string tension in lattice qcd. *Phys. Lett.*, 192B:163, 1987.
20. M. Lüscher and P. Weisz. Definition and general properties of the transfer matrix in continuum limit improved lattice gauge theories. *Nucl. Phys.*, B240:349, 1984.
21. N. A. Campbell et al. Heavy quark potentials and hybrid mesons from su(3) lattice gauge theory. *Nucl. Phys.*, B306:51, 1988.
22. R. Sommer. A new way to set the energy scale in lattice gauge theories and its applications to the static force and alpha-s in su(2) yang-mills theory. *Nucl. Phys.*, B411:839, 1994.
23. F. Niedermayer et al. Fixed point gauge actions with fat links: Scaling and glueballs. *Nucl. Phys.*, B597:413–450, 2001.
24. Colin J. Morningstar and Mike J. Peardon. The glueball spectrum from an anisotropic lattice study. *Phys. Rev.*, D60:034509, 1999.
25. A. Vaccarino and D. Weingarten. Glueball mass predictions of the valence approximation to lattice qcd. *Phys. Rev.*, D60:114501, 1999.
26. Michael J. Teper. Glueball masses and other physical properties of su(n) gauge theories in d = (3+1): A review of lattice results for theorists. 1998.